Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(99)00038-6
J. Appl. Maths Mechs, Vol. 63, No. 2, pp. 291–298, 1999

0021-8928/99/$—see front matter

AN ASYMPTOTIC METHOD OF SOLVING TRANSIENT DYNAMIC CONTACT PROBLEMS†

V. B. ZELENTSOV

Rostov-on-Don

(*Received* 13 *August* 1998)

Using a special approximation in the complex plane of the symbol of the kernel of the contact-problem integral equation, an asymptotic form of its solution is constructed which is the fundamental solution of the transient dynamic plane contact problem of the impact of a rigid punch with an elastic half-plane for short interaction times. The proposed approximation of the kernel symbol enables it to be approximated in the complex plane with any previously specified accuracy. Unlike existing approaches [1, 2, etc.], the approximation of the kernel symbol of the integral equation employed here enables the solution of this problem to be obtained in the form of simple formulae not containing singular quadratures.

1. FORMULATION OF THE PROBLEM AND ITS INTEGRAL EQUATION

Consider the plane contact problem of the impact of a punch of width $2a$ ($|x| \leqslant a$) with an elastic half-plane ($y \geqslant 0$, $|x| < \infty$) with initial impression velocity v_0 and ignoring friction forces in the contact region. The shape of the punch and its law of motion in the elastic medium are defined by the function $\varepsilon(x, t)(|x| \leqslant a, t \geqslant 0)$. At the initial instant, taking into account the fact that before impression the elastic medium is at rest, the displacements of the elastic medium $u = u(x, y, t)$ and $\upsilon = \upsilon(x, y, t)$ and their velocities are assumed to be zero.

In the generally accepted notation of the theory of elasticity [2], the mixed boundary conditions of the contact problem when $y = 0$ ($t > 0$) have the form

$$\begin{aligned} &\tau_{xy} = 0, && -\infty < x < \infty \\ &\sigma_{yy} = 0, && -\infty < x < -a,\ a < x < \infty \\ &v = \varepsilon(x, t), && -a < x < a \end{aligned} \tag{1.1}$$

with the condition that at infinity ($\sqrt{(x^2 + y^2)} \to \infty$) the displacements u and υ, together with their partial derivatives with respect to x and y, vanish.

Using Laplace integral transformations with respect to time t [3] and a Fourier integral transformation with respect to the longitudinal coordinate x [4], applied to the differential equations of the theory of elasticity [2] and to the mixed boundary conditions (1.1), taking into account the initial conditions and the conditions at infinity, the solution of the contact problem can be reduced to the following integral equation

$$\int_{-a}^{a} \varphi^L(\xi, p) k(\xi - x, p) d\xi = 2\pi\varepsilon^L(x, p), \quad |x| \leqslant a \tag{1.2}$$

$$k(t, p) = \int_{-\infty}^{\infty} K(\alpha, p) e^{i\alpha t} d\alpha, \quad K(\alpha, p) = \frac{\sigma_2(\sigma_1^2 - \alpha^2)}{R(\alpha, p)}$$

$$R(\alpha, p) = (\sigma_1^2 + \alpha^2)((\lambda + 2\mu)\sigma_1^2 - \lambda\alpha^2) - 4\mu\alpha^2\sigma_1\sigma_2$$

$$\sigma_1 = (\alpha^2 + p^2 / c_2^2)^{1/2}, \quad \sigma_1 = (\alpha^2 + p^2 / c_1^2)^{1/2}, \quad c_1 = [(\lambda + 2\mu)/\rho]^{1/2}$$

$$c_2 = (\mu/\rho)^{1/2}$$

with respect to the unknown transformant $\varphi^L(x, p)$ of the contract stresses, $\varphi(x, t)$, which occur under the punch, and $\sigma_{yy}(x, 0, t) = \varphi(x, t)$. Here ρ is the density of the material of the half-plane, λ and μ are the Lamé elastic constants [2] and $\varepsilon^L(x, p)$ is the Laplace transform of the function $\varepsilon(x, t)$.

†*Prikl. Mat. Mekh.* Vol. 63, No. 2, pp. 303–311, 1999.

The following expressions are obtained for the Laplace transform of the vertical displacements $v^L(x, y, p)$ and of the normal stresses $\sigma^L_{yy}(x, y, p)$ ($y \geqslant 0$, $-\infty < x < \infty$)

$$v^L(x,y,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi^{LF}(\alpha,p)\left[-2\alpha^2\sigma_2 e^{-\sigma_1 y} + \left(\sigma_1^2+\alpha^2\right)\sigma_2 e^{-\sigma_2 y}\right]\frac{e^{-i\alpha x}}{R(\alpha,p)}d\alpha \tag{1.3}$$

$$\sigma^L_{yy}(x,y,p) = \frac{2\mu i}{\pi}\int_{-\infty}^{\infty} \varphi^{LF}(\alpha,p)\left[-2\alpha^2\sigma_1\sigma_2 e^{-\sigma_1 y} + \left(\sigma_1^2+\alpha^2\right)((\lambda+2\mu)\sigma_2^2 - \right.$$

$$\left. -\lambda\alpha^2)e^{-\sigma_2 y}\right]\frac{e^{-i\alpha x}}{R(\alpha,p)}d\alpha$$

In Eq. (1.2) we have introduced the following notation (ν is Poisson's ratio)

$$p = c_2 p', \quad \beta^2 = c_2^2 / c_1^2 = (1-2\nu)/[2(1-\nu)]$$

In the inner integral of integral equation (1.2) we have made the substitution $\alpha = p'u$, and in the outer integral we have made the substitutions $\xi = a\xi'$ and $x = ax'$. As a result of these changes, integral equation (1.2) reduces to the dimensionless form (the prime is omitted)

$$\int_{-1}^{1} \varphi^L(\xi,p)k\left(\frac{\xi - x}{\Lambda}\right)d\xi = 2\pi f^L(x,p), \quad |x| \leqslant 1 \tag{1.4}$$

$$k(t) = \int_\Gamma K(u)e^{iut}du, \quad K(u) = 2(1-\beta^2)\frac{\sqrt{u^2+\beta^2}}{R_0(u)}$$

$$R_0(u) = (2u^2+1)^2 - 4u^2\sqrt{u^2+1}\sqrt{u^2+\beta^2}$$

$$f^L(x,p) = \frac{2}{a}(1-\beta^2)\mu\varepsilon^L(x,p), \quad \Lambda = \frac{c_2}{ap}$$

Equation (1.4) was then multiplied by $2(1 - \beta^2)$, and the contour of integration Γ in the complex plane $u = \sigma + i\tau$ makes an angle of $-\arg p$ with the real axis ($\tau = 0$).

2. THE ASYMPTOTIC SOLUTION OF THE INTEGRAL EQUATION FOR LARGE p

The symbol of the kernel of integral equation (1.4), the function $K(u)$, possesses the following properties: it is even with respect to u, real on the real axis of the complex plane $u = \sigma + i\tau$, and the behaviour of $K(u)$ at zero and at infinity is given by the relations

$$K(u) = |u|^{-1} + O(|u|^{-3}), \quad |u| \to \infty \tag{2.1}$$

$$K(u) = K(0) + O(u^2), \quad u \to 0; \quad K(0) = 2\beta(1-\beta^2) \tag{2.2}$$

In the complex plane $u = \sigma + i\tau$ the function $K(u)$ has four branching points $u = \pm i\beta$ and $u = \pm i$, and two poles $u = \pm i\eta_0$ (Rayleigh poles) [5].

For a unique representation of the function $K(u)$ in the complex plane u we make cuts which pass from the branching points $u = i$, $u = i\beta$ to $i\infty$ along the positive part ($\operatorname{Im} u \geqslant 0$) of the imaginary axis and from the branching points $u = -i$ and $u = -i\beta$ to $-i\infty$ along the negative part ($\operatorname{Im} u \leqslant 0$) of the imaginary axis. In the plane cut in this way with deleted Rayleigh poles $u = \pm i\eta_0$, the function $K(u)$ is analytic, including the poles $|\operatorname{Im}(u)| < \beta$, $\beta < 1 < \eta_0$.

To construct the zeroth term of the asymptotic form of the solution of integral equation (1.4) for large values of p, it is sufficient to construct the zeroth term of the asymptotic form of the solution of (1.4) for small values of Λ. After deforming the contour of integration Γ in the complex plane $u = \sigma + i\tau$ into a contour parallel to the real axis ($\tau = 0$) and arranged in the strip $|\operatorname{Im}(u) < \beta| < \beta$, the zeroth term of the asymptotic form of the solution of integral equation (1.4) for small Λ can be represented in the form of the superposition of solutions of the following integral equations [6]

$$\begin{aligned}&\int_{-1}^{\infty}\varphi_+^L(\xi,p)k\left(\frac{\xi-x}{\Lambda}\right)d\xi=2\pi f^L(x,p),\quad -1\leqslant x<\infty\\&\int_{-1}^{1}\varphi_-^L(\xi,p)k\left(\frac{\xi-x}{\Lambda}\right)d\xi=2\pi f^L(x,p),\quad -\infty\leqslant x\leqslant 1\\&\int_{-\infty}^{\infty}\varphi_\infty^L(\xi,p)k\left(\frac{\xi-x}{\Lambda}\right)d\xi=2\pi f^L(x,p),\quad -\infty<x<\infty\end{aligned} \tag{2.3}$$

$$k(t)=\int^{\infty}K(u)e^{iut}du$$

using the formula

$$\varphi^L(x,p)=\varphi_+^L\left(\frac{1+x}{\Lambda},p\right)+\varphi_-^L\left(\frac{1-x}{\Lambda},p\right)-\varphi_\infty^L\left(\frac{x}{\Lambda},p\right) \tag{2.4}$$

For this purpose, we have made the replacement of variables $\xi = \Lambda\xi' - 1, x = \Lambda x' - 1$ and $\xi = 1 - \Lambda\xi', x = 1 - \Lambda x'$, respectively, in the first two equations of (2.3), while in the third equation of (2.3) we have made the replacement $\xi = \Lambda\xi', x = \Lambda x'$.

As a result of these changes, Eqs (2.3) take the form (we have omitted the primes)

$$\int_0^{\infty}\varphi_\pm^L(\xi,p)k(\xi-x)d\xi=2\pi f^L(\pm\Lambda x\mp 1,p)\Lambda^{-1},\quad 0\leqslant x<\infty \tag{2.5}$$

$$\int_{-\infty}^{\infty}\varphi_\infty^L(\xi,p)k(\xi-x)d\xi=2\pi f^L(\Lambda x,p)\Lambda^{-1},\quad -\infty<x<\infty \tag{2.6}$$

Equations (2.5) are the Wiener–Hopf equations on the half-axis, while (2.6) is the equation of convolution on the axis [7].

The solution of integral equation (2.6) is obtained by applying a Fourier integral transformation to it and is given by the formula

$$\varphi_\infty^L(x,p)=\frac{1}{2\pi\Lambda}\int_{-\infty}^{\infty}\frac{f^{LF}(u,p)\exp(-iux)}{K(u)}du \tag{2.7}$$

The solution of integral equation (2.5) is obtained by applying a standard Wiener–Hopf procedure [9, 10] to it. The solution of the first integral equation of (2.5) can thereby be represented by the formula

$$\varphi_+^L(x,p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{g_+(u)\exp(-iux)}{K_+(u)}du \tag{2.8}$$

where $g_+(u)$ is a function that is regular in the upper half-plane ($\operatorname{Im} u > \tau_-, 0 \leqslant |\tau_-| \leqslant \beta$) and is defined from the relation

$$g_+(u)+g_-(u)=\frac{F_+(u)}{\Lambda K_-(u)};\quad F_+(u)=\int_0^{\infty}f^L(\Lambda\xi-1,p)\exp(iu\xi)d\xi \tag{2.9}$$

while the functions $K_+(u)$ and $K_-(u)$ are found by factorizing the function $K(u) = K_+(u)K_-(u)$; they are regular in the upper half-plane ($\operatorname{Im} u > \tau_-, \tau_- \leqslant 0$) and the lower half-plane ($\operatorname{Im} u < \tau_+, \tau_+ \geqslant 0$) of the complex plane $u = \sigma + i\tau$, respectively.

The solution $\varphi_-^L(x, p)$ of Eqs (2.5) is given by (2.8), in which $F_+(u)$ is given by (2.8) and (2.9), but with the integrand replaced by $f^L(-\Lambda\xi + 1, p)\exp(iu\xi)$ in the formula for $F_+(u)$.

After calculating the quadratures (2.7) and (2.8) and returning to the old variables, the zeroth term of the asymptotic solution of (1.4) is given by (2.4). In general, it is difficult to calculate quadratures of the type (2.8) in analytic form since $K_+(u)$ and $K_-(u)$ are given in singular quadratures [2] as a result of the factorization of $K(u)$.

3. APPROXIMATION OF THE FUNCTION $K(u)$ AND ITS FACTORIZATION

To obtain an approximate solution of the integral equation of dynamic stationary and static mixed problems we used a method based on a special approximation of the symbol of the kernel $K(u)$ of the integral equation along its axis of integration [6, 8–11, etc.]. We chose as the approximating function a function that could be factorized by elementary means and which enabled quadratures of the type (2.8) to be calculated in analytical form. It has been established [10, 11], that, by an appropriate choice of the approximating function, the error of the approximate solution of the integral equation does not exceed the approximation errors.

When solving transient dynamic contact problems the need arises to choose a form of the approximating function in the complex plane which will satisfy the above requirements and enable the physical meaning of the problem to be preserved in the solution. The usual methods of approximating the function in the complex plane, for example, the Padé approximation [12], do not satisfy this requirement.

Here we will take as the approximation of the functions $K(u)$, which satisfies all the above requirements, a function $K_0(u)$ of the following form

$$K_0(u) = \frac{\sqrt{u^2+\beta^2}}{u^2+\eta_0^2} M_n(u) \tag{3.1}$$

$$M_n(u) = \exp\left[\frac{1}{2}\sum_{k=0}^{n} d_k((\sqrt{\beta+iu}-\sqrt{1+iu})^{2k+2} + (\sqrt{\beta-iu}-\sqrt{1-iu})^{2k+2})\right]$$

The constants d_k are found from the conditions for best approximation of $K(u)$ in the complex plane $u = \sigma + i\tau$. The Rayleigh poles $\pm i\eta_0$ of the function $K(u)$ are found from the equation $R_0(u) = 0$.

The function $K_0(u)$ in (3.1) is factorized, i.e. it is represented in the complex plane in the form $K_0(u) = K_+^0(u)K_-^0(u)$, by elementary methods, and we then have

$$K_\pm^0(u) = \frac{\sqrt{\beta \mp iu}}{\eta_0 \mp iu} \exp\left[\frac{1}{2}\sum_{k=0}^{n} d_k(\sqrt{\beta\mp iu}-\sqrt{1\mp iu})^{2k+2}\right] \tag{3.2}$$

The functions $K_+^0(u)$ possess the property

$$K_+^0(u) = K_-^0(-u) \tag{3.3}$$

and are regular in the half-planes $\mathrm{Im}(u) > -\beta$ and $\mathrm{Im}(u) < \beta$ ($\beta > 0$), respectively, with asymptotic forms

$$K_\pm^0(u) = \frac{1}{\sqrt{-iu}} + O\left(\frac{1}{|u|}\right), \quad |u| \to \infty \tag{3.4}$$

$$K_\pm^0(u) = \sqrt{K(0)} + O(u), \quad |u| \to 0 \tag{3.5}$$

Note that the form of approximation (3.1) is not unique.

4. THE ASYMPTOTIC SOLUTION OF INTEGRAL EQUATION (1.4) WITH THE APPROXIMATED KERNEL

The solutions $\varphi_\pm^L(x,p)$ of the Wiener–Hopf integral equation (2.5) when the symbol of the kernel $K(u)$ of this equation is replaced by the approximating function $K_0(u)$ is given by the general formula (2.8), in which it is sufficient to substitute $K_\pm^0(u)$, defined by (3.2), instead of $K_\pm(u)$. The zeroth term of the asymptotic solution of integral equation (1.4) is given by (2.4), after changing in $\varphi_\pm^L(x,p)$, $\varphi_\infty^L(x, p)$ to the old variables.

For the case of a plane punch, when $\varepsilon(x,t) = \varepsilon(t)$ and the right-hand side of integral equation (1.4) takes the form

$$f^L(x,p) = \frac{2}{a}(1-\beta^2)\mu\varepsilon^L(p)$$

where $\varepsilon^L(p)$ is the Laplace transform of the function $\varepsilon(t)$, the solution $\varphi_\infty^L(x,p)$ of integral equation (2.6) can be represented by the formula

$$\varphi_\infty^L(x,p)=\frac{\theta\varepsilon^L(p)}{\Lambda K(0)},\quad \theta=\frac{2}{a}(1-\beta^2)\mu \tag{4.1}$$

The solutions of integral equation (2.5) in this case are given by the formula

$$\varphi_\pm^L(x,p)=\frac{\theta\varepsilon^L(p)}{2\pi\Lambda K_-^0(0)}\int_{-\infty}^{\infty}\frac{\exp(-iux)}{-iuK_+^0(u)}du \tag{4.2}$$

When calculating the quadratures in (4.2) we make the replacement of variable $-iu = s$. In the complex plane $s = u + i\upsilon$ the integrand $[sK_+^0(is)]^{-1}$ has singular points: at zero ($s = 0$)—a first-order pole and two algebraic-type branching points when $s = -\beta$ and $s = -1$. For a unique representation of the integrand of (4.2) in the complex plane s we make cuts from $s = -\beta$ and $s = -1$ to $-\infty$ along the negative part of the real axis, with a subsequent choice of the branches of the function $\sqrt{(\beta + s)}$ and $\sqrt{(1 + s)}$ with the condition $\sqrt{1} = 1$. Evaluation of the integral in (4.2) leads to the formula

$$\varphi_\pm^L(x,p)=\frac{\theta\varepsilon^L(p)}{\pi\Lambda K_-^0(0)}\left[\int_1^\infty q(\chi_1(y),y)dy+\int_\beta^1 q(P_n(y),y)\cos[\zeta(y)Q_{n-1}(y)]dy+\frac{\pi}{\sqrt{K(0)}}\right] \tag{4.3}$$

$$q(w,y)=\vartheta(y)\exp(-w-yx),\quad \zeta(y)=\sqrt{y-\beta}\sqrt{1-y}$$

$$\chi_1(y)=\frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}d_k\left(\sqrt{y-\beta}-\sqrt{y-1}\right)^{2k+2}$$

$$\vartheta(y)=\frac{y-\eta_0}{y\sqrt{y-\eta_0}},\quad Q_{n-1}(y)=\frac{1}{\zeta(y)}\operatorname{Im}(-\chi_2(y)),\quad P_n(y)=\operatorname{Re}[-\chi_2(y)]$$

$$\chi_2(y)=\frac{1}{2}\sum_{k=0}^{n}d_k\left(i\sqrt{y-\beta}-\sqrt{1-y}\right)^{2k+2}$$

When returning to the old variables in (4.1) and (4.3) we take into account the fact that in this case the zeroth term of the asymptotic form of the solution of integral equation (1.4) is represented by formula (2.4).

5. THE SOLUTION OF THE CONTACT PROBLEM FOR SMALL t

The asymptotic solution of the contact problem in question for a plane punch for small t is obtained by changing to the originals of the Laplace transformation in (2.4), (4.1) and (4.3) (the solutions of integral equation (1.4)). The originals of the functions $\varphi_\pm^L(x,p)$, $\varphi_\infty^L(x,p)$, after reverting to the dimensional variable x in (4.1) and (4.3), are given by the formulae [13]

$$\varphi_\infty(x,t)=\frac{2(1-\beta^2)\mu}{c_2K(0)}E(t) \tag{5.1}$$

$$\varphi_\pm(a\pm x,t)=\frac{2(1-\beta^2)\mu}{\pi K_-^0(0)\sqrt{c_2(a\pm x)}}\left[\sum_{k=1}^{2}\frac{\partial}{\partial t}\int_0^t f_k(\tau,x)\varepsilon(t-\tau)d\tau+b\sqrt{a\pm x}E(t)\right] \tag{5.2}$$

$$E(t)=\varepsilon'(t)+\varepsilon(0)\delta(t)$$

$$f_1(t,x)=H(t-t_2^\pm)\varkappa(t)\exp(-\psi_1(t,x))$$

$$f_2(t,x)=[H(t-t_1^\pm)-H(t-t_2^\pm)]\varkappa(t)\exp(-P_n(t/t_2^\pm))\cos(\psi_2(t,x))$$

$$\varkappa(t)=\frac{t-t_R^\pm}{t\sqrt{t-t_1^\pm}}$$

$$\psi_1(t,x)=-\frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1}d_k\left(\frac{\sqrt{t-t_1^\pm}-\sqrt{t-t_2^\pm}}{t_2^\pm}\right)^{2k+2},\quad \psi_2(t,x)=\frac{\sqrt{t-t_1^\pm}-\sqrt{t_2^\pm-t}}{t_2^\pm}Q_{n-1}\left(\frac{t}{t_2^\pm}\right)$$

$$t_i^{\pm}=\frac{a\pm x}{c_i}\ \ (i=1,2),\quad t_R^{\pm}=\eta_0 t_2^{\pm},\quad b=\frac{\pi K_-^0(0)}{c_2 K(0)}$$

where $H(t)$ is the Heaviside function, $\delta(t)$ is the Dirac delta function, $P_n(t)$ and $Q_{n-1}(t)$ are defined in (4.3), and $\varepsilon(0)$ is the initial impression of the punch (before $t = 0$). The zeroth term of the asymptotic solution of the contact problem is defined by the formula

$$\varphi(x,t)=\varphi_+(a+x,t)+\varphi_-(a-x,t)-\varphi_\infty(x,t) \tag{5.3}$$

Formulae (5.1)–(5.3) enable us to analyse the dynamics of the contact stresses $\varphi(x, t)$. The following is established by such an analysis: (1) the contact stresses are proportional to the rate of impression of the punch $\varepsilon'(t)$ until the arrival of waves from the edges of the punch (for all $x \in (c_1 t - a, a - c_1 t)$), and then are added to them; (2) the contact stresses contain fixed singularities of the form $(a + x)^{-1/2}$ which arise at the edges of the punch ($x = \pm a$), and also mobile singularities at the wave fronts of the longitudinal waves ($\varepsilon(0) \neq 0$), which propagate from the edges of the punch with velocity c_1, of the form $(c_1 t - (a \pm x))^{-1/2}$, whereas a wave front of the transverse wave, moving with velocity $c_2 (c_2 < c_1)$, has no singularities [1, 2].

In the special case when the punch is simultaneously impressed into the elastic medium at the initial instant $t = 0$ (in this case $\varepsilon(t) = \varepsilon_0(H(t))$, the contact stresses are given by (5.3), in which

$$\varphi_\infty(x,t)=\frac{2(1-\beta^2)\mu\varepsilon_0}{c_2K(0)}\delta(t) \tag{5.4}$$

$$\varphi_\pm(a\pm x,t)=\frac{2(1-\beta^2)\mu\varepsilon_0}{\pi K_-^0(0)\sqrt{c_2(a\pm x)}}\left[\sum_{k=1}^{2}f_k(t,x)+b\sqrt{a\pm x}\,\delta(t)\right] \tag{5.5}$$

while $f_k(t, x)$ ($k = 1, 2$) and b are given in (5.2).

In the case of the approximation of $K_0(u)$ of the form (3.1) for $n = 0$ in the case considered the formulae for solving contact problem (5.3)–(5.5) take the simplest form, since they do not contain quadratures, while $f_x(t, x)$ are given by the formulae

$$f_1(t,x)=H(t-t_2^{\pm})\varkappa(t)\exp\left[\frac{d_0(1-\beta^2)t_2^{\pm}}{2\left(\sqrt{t-t_1^{\pm}}+\sqrt{t-t_2^{\pm}}\right)^2}\right]$$

$$f_2(t,x)=[H(t-t_1^{\pm})-H(t-t_2^{\pm})]\varkappa(t)\exp\left[\left(-\frac{1+\beta}{2}+\frac{t}{t_2^{\pm}}\right)d_0\right]\cos(\psi_2(t,x))$$

$$\psi_2(t,x)=\frac{d_0\sqrt{t-t_1^{\pm}}\sqrt{t_2^{\pm}-t}}{t_2^{\pm}}$$

The function $\varkappa(t)$ is given in (5.2).

If the constant of the approximation d_0 is determined from the condition $K(0) = K_0(0)$, we have

$$d_0=\ln(\eta_0^2K(0)\beta^{-1})/(1-\sqrt{\beta})^2$$

The error of this approximation ($n = 0$) for all $\nu \in [0; 0.44]$ along the real axis ($\tau = 0$) of the complex plane $u = \sigma + i\tau$ does not exceed 4%, while over the whole range $\nu \in [0; 0.5]$ it does not exceed 22%. The increase in the approximation error as $\nu \to 0.5$ is due to the fact that the material of the half-plane becomes incompressible and $K(u)$ takes a qualitatively different mathematical form when $\nu = 0.5$: the analytic expression for the function contains only one algebraic root.

The formulae obtained for the contact stresses enable one to construct the wave field of the displacements and stresses in an elastic medium. To do this one can use the Cagniard–de Hoope method [2, 14] to evaluate the integrals in (1.3). The formula for representing the wave field of the normal stresses $\sigma_{yy}(x, y, t)$, after using the Cagniard–de Hoop method to evaluate the integral in (1.3), takes the form

$$\sigma_{yy} = \frac{\mu}{c_2\beta}\varepsilon'\left(t - \frac{y}{c_1}\right) + \frac{4(1-\beta^2)\mu}{c_2 K_-^0(0)} \sum_{k=1}^{2} \frac{\partial}{\partial t} \int_{t_k^\pm}^{t} H_k^\pm(\tau, x)\varepsilon(\tau - t)d\tau \tag{5.6}$$

and contains four integrals, since $H_k^\pm$ denotes that either H_k^+ or H_k^- is taken alternately. We have introduced the following notation

$$H_1^\pm(t,x) = \mathrm{Im}\left[\mp\frac{1}{sK_+^0(\pm is)}\frac{(s^2 - \tfrac{1}{2})^2}{R(s)}\frac{ds}{d\tau}\right]\Bigg|_{s=S_1^\pm}, \quad S_1^\pm = \begin{cases}\beta(\tau c_1(a\pm x) - y\rho_{11}^\mp)r_\pm^{-2}, & y \leq \tau c_1 \leq r_\pm \\ \beta(\tau c_1(a\pm x) + iy\rho_{12}^\pm)r_\pm^{-2}, & \tau c_1 \geq r_\pm\end{cases}$$

$$H_2^\pm(t,x) = \mathrm{Im}\left[\mp\frac{1}{K_+^0(\pm is)}\frac{\sigma_1\sigma_2}{R(s)}\frac{ds}{d\tau}\right]\Bigg|_{s=S_2^\pm}, \quad S_2^\pm = \begin{cases}(\tau c_2(a\pm x) - y\rho_{21}^\mp)r_\pm^{-2}, & y \leq \tau c_2 \leq r_\pm \\ (\tau c_2(a\pm x) + iy\rho_{22}^\pm)r_\pm^{-2}, & \tau c_2 \geq r_\pm\end{cases}$$

$$\rho_{mi}^\pm = ((-1)^{i+1}(r_\pm^2 - \tau^2 c_m^2))^{1/2}, \quad m, i = 1,2; \quad r_\pm = ((a\pm x)^2 + y^2)^{1/2}$$

$$R(s) = (s^2 - \tfrac{1}{2})^2 + s^2\sigma_1\sigma_2, \quad \sigma_1 = \sqrt{1-s^2}, \quad \sigma_2 = \sqrt{\beta^2 - s^2}$$

$$t_1^\pm = \frac{r_\pm}{c_1}, \quad t_2^\pm = \begin{cases} r_\pm / c_2, & \cos\theta < \beta \\ -\beta(a\pm x) - y\sqrt{1-\beta^2}, & \cos\theta > \beta\end{cases}, \quad \theta = \mathrm{arctg}\frac{y}{x}$$

The function $K_+^0(u)$ is given by (3.5) for the general case of the approximation. Formula (5.6) enables us to give a geometrical picture of the stress wave field $\sigma_{yy}(x, y, t)$ and to indicate features on the wave fronts of elastic waves in an elastic half-plane [2].

6. THE MOTION OF A PUNCH IN AN ELASTIC MEDIUM

The impression of a plane rigid punch $\varepsilon(t)$ into an elastic half-plane can be determined from the differential equation (the punch is represented by a point mass M), with initial conditions

$$M\ddot{\varepsilon}(t) = Q(t); \quad \varepsilon(0) = \varepsilon_0, \quad \dot{\varepsilon}(0) = v_0 \tag{6.1}$$

where $Q(t)$ is the elastic resistance force of the medium.

To determine the Laplace transform of the elastic resistance force of the medium

$$Q^L(p) = -\int_{-a}^{a}\varphi^L(x,p)dx$$

we will use the zeroth term of the asymptotic form $\varphi^L(x, p)$ of the solution of integral equation (1.4). To do this we take the asymptotic solution of the integral equation for small t in the new multiplicative form [6]

$$\varphi^L(x,p) = \varphi_+^L(a+x,p)\varphi_-^L(a-x,p)/\varphi_\infty^L(x,p) \tag{6.2}$$

the realization of which leads to the formula

$$Q^L(p) = -2(1-\beta^2)\mu a\frac{K(0)}{K_-(0)}\frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c}\omega^2(iu)\exp(\gamma u)du \tag{6.3}$$

$$\omega(iu) = -(uK_+^0(iu))^{-1}, \quad \gamma = 2/\Lambda, \quad \mathrm{Re}\, c > 0$$

The functions $K_\pm^0(u)$ are given by (3.5).

For $t < 2a/c_1$, formula (6.3) takes the form

$$Q^L(p) = -\frac{2(1-\beta^2)\mu K(0)}{K_-(0)}\left[\frac{2ap}{c_2K_+^2(0)} - \frac{2iK_+'(0)}{K_+^3(0)}\right]\varepsilon^L(p) \tag{6.4}$$

From the solution of (6.1) we obtain, by means of a Laplace transformation using expression (6.4)

Table 1

No.	Material	$\mu \times 10^{10}$, N/m	$\rho \times 10^3$, kg/m^2	ν	max ε, mm	$t_* \times 10^4$, s
1	Aluminium	2.5	2.7	0.35	2.16	1.21
2	Granite	4.0	3.0	0.10	1.15	0.62
3	Copper	3.0	8.9	0.35	1.82	1.06
4	Steel	8.0	7.7	0.25	1.24	0.71
5	Glass	2.9	2.5	0.20	2.13	1.17
6	Cast iron	4.4	7.0	0.25	1.60	0.90

$$\varepsilon^L(p) = \frac{\varepsilon_0 p + v_0}{(p+u_*)^2 + \delta_*}, \quad u_* = \frac{\mu a}{\beta c_2 M} \tag{6.5}$$

$$\delta_* = \frac{\mu\zeta_0}{M\beta^2\eta_0} - u_*^2, \quad \zeta_0 = \sqrt{\beta}\eta_0 \ln[2(1-\beta^2)\eta_0^2] + 2\beta - \eta_0$$

Calculations show that $\delta_* > 0$ for all $\nu \in [0; 0.44]$, i.e. for those values of ν for which the approximation $K_0(u)$ given by (3.1) for $n = 0$ allows of an error of less than 4% for $K(u)$ along the real axis. For such values of ν, the value of the impression of the punch takes the form ($\varepsilon_0 = 0$)

$$\varepsilon(t) = \frac{v_0}{\sqrt{\delta_*}} \exp(-u_* t) \sin\sqrt{\delta_* t} \tag{6.6}$$

The depth of maximum impression of the punch into the elastic medium is then given by the formula

$$\max \varepsilon(t_*) = \frac{v_0\theta_*}{\sqrt{\delta_*(1+\theta_*)}} \exp\left(-\frac{\operatorname{arctg}\theta_*}{\theta_*}\right); \tag{6.7}$$

$$t_* = \frac{1}{\sqrt{\delta_*}} \operatorname{arctg}\frac{\sqrt{\delta_*}}{u_*}, \quad \theta_* = \frac{u_*}{\sqrt{\delta_*}}$$

where t_* is the time of maximum impression, found from the condition $\varepsilon(t_*) = 0$.

Using (6.7) we calculated maxε and t_* for $M = 200$ kg, $a = 1$ cm, and $v_0 = 30$ m/s. The results are given in Table 1 for various materials with an indication of their characteristics [14], used in the calculations.

I wish to thank V. M. Aleksandrov for his interest.

REFERENCES

1. FLITMAN, L. M., The dynamic problem of a punch on an elastic half-plane. *Prikl. Mat. Mekh.*, 1959, **23**, 4, 697–705.
2. PARTON, V. Z. and PERLIN, P. I., *Methods of the Mathematical Theory of Elasticity*. Nauka, Moscow, 1981.
3. DITKIN, V. A. and PRUDNIKOV, A. P., *The Operational Calculus*. Vyssh. Shkola, Moscow, 1966.
4. SNEDDON, I., *Fourier Transforms*. McGraw-Hill, New York, 1951.
5. SEIMOV, V. M., *Dynamic Contact Problems*. Naukova Dumka, Kiev, 1976.
6. VOROVICH, I. I., ALEKSANDROV, V. M. and BABESHKO, V. A., *Non-classical Mixed Problems of the Theory of Elasticity*. Nauka, Moscow, 1974.
7. ZABREIKO, P. P., KOSHELEV, A. I., KRASNOSEL'SKII, M. A. *et al.*, *Integral Equations*. Nauka, Moscow, 1968.
8. NOBLE, B., *Methods Based on the Wiener–Hopf Technique*. Pergamon Press, Oxford, 1958.
9. KOITER, W. T., Approximate solution of Wiener–Hopf type integral equation with applications. Pts I–III. *Proc. Koninkl. Ned. Akad. Wetenchap.*, 1954, **57**, 558–559.
10. VOROVICH, I. I. and BABESHKO, V. A., *Dynamic Mixed Problems of the Theory of Elasticity for Non-classical Domains*. Nauka, Moscow, 1979.
11. VOYEV, S. I. and SUMBATYAN, M. A., The dynamic contact problem for an elastic half-plane at high vibration frequencies. *Prikl. Mat. Mekh.*, 1985, **49**, 6, 1039–1043.
12. BAKER, G. and GRAVES-MORRIS, P., *Padé Approximants*. Addison-Wesley, London, 1981.
13. BATEMAN, H. and ERDÉLYI, A., *Tables of Integral Transforms*. McGraw-Hill, New York, 1954.
14. DE HOOP, A. T., A modification of Cagniard's method for solving seismic pulse problems. *Appl. Sci. Res. Sec. B.*, 1960, **8**, 4, 349–356.
15. KOSHKIN, N. I. and SHIRKEVICH, M. G., *Handbook of Elementary Physics*. Nauka, Moscow, 1965.

Translated by R.C.G.